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Characteristic classes and transversality

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Abstract

Let ξ be a smooth vector bundle over a differentiable manifold M. Let $h:\varepsilon^{n-i+1}\to\xi$ be a generic bundle morphism from the trivial bundle of rank $n-i+1$ to ξ. We give a geometric construction of the Stiefel–Whitney classes when ξ is a real vector bundle, and of the Chern classes when ξ is a complex vector bundle. Using h we define a differentiable closed manifold $\tilde{Z}(h)$ and a map $\phi:\tilde{Z}(h)\to M$ whose image is the singular set of h. The ith characteristic class of ξ is the Poincaré dual of the image, under the homomorphism induced in homology by ϕ, of the fundamental class of the manifold $\tilde{Z}(h)$. We extend this definition for vector bundles over a paracompact space, using that the universal bundle is filtered by smooth vector bundles.
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1. Introduction

The theory of characteristic classes began with the works of Hassler Whitney [1] and Eduard Stiefel [2] in 1935, Lev Pontrjagin in 1942 [3] and Shing-Shen Chern in 1946 [4]. Given a vector bundle ξ of rank n over a space B the problem is to determine how many linearly independent sections could be defined on ξ. For instance, ξ is trivial if and only if it admits n linearly independent sections. If B is a CW complex then the ith Stiefel–Whitney class $w_i(\xi)$ of a real vector bundle ξ is defined as the primary obstruction to the existence of $n-i+1$ linearly independent sections over the ith skeleton of B. This implies that if ξ has $n-i+1$ linearly independent sections, then $w_i(\xi)=0$. Analogously, the ith Chern class $c_i(\xi)$ of a complex vector bundle ξ is the obstruction to the existence of a complex $n-i+1$-frame in ξ. This method to define characteristic classes uses tools from algebraic topology, in particular the obstruction classes belong to a cohomology group with local coefficients.

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There are other methods to define the characteristic classes: using cohomology operations [5], computing the cohomology of classifying spaces [6] or computing the cohomology of the projective bundle associated to ξ [7]. All these methods are of algebraic nature. There is another method for smooth bundles known as Chern–Weil theory which uses a connection on ξ [5, Appendix C].

The aim of the present paper is to give a geometric construction of characteristic classes using essentially transversality and following the idea of the original definition. This construction was inspired by a remark made by Atiyah in the introduction of [8].

Let ξ be a smooth vector bundle over a differentiable manifold M. The existence of $n-i+1$ linearly independent cross-sections of ξ is equivalent to the existence of a bundle morphism $h:\varepsilon^{n-i+1}\to\xi$ from the product bundle ε^{n-i+1} to ξ such that h is injective in each fibre. With this formulation the obstruction to the existence of the $n-i+1$ linearly independent cross-sections is represented by the subset $\bar{Z}(h)$ of points of M where h is not injective. In general $\bar{Z}(h)$ is not a manifold, but if h is generic, it is a stratified manifold [9]. Using a construction by Porteous [10] we define a differentiable closed manifold $\tilde{Z}(h)$ and a map $\phi:\tilde{Z}(h)\to M$ whose image is $\bar{Z}(h)$. The image of the fundamental class of $\tilde{Z}(h)$ under the homomorphism induced in homology by ϕ gives an element in the homology of M and we define the ith characteristic class $\mathbf{Cl}_i(\xi)$ of ξ as its Poincaré dual. The main result of the paper (Theorem 11) proves that such classes satisfy a set of axioms which are equivalent to the axioms that define the Stiefel–Whitney or Chern classes given by Hirzebruch in [11].

In Theorem 11 the construction of the classes $\mathbf{Cl}_i(\xi)$ is done only for smooth vector bundles over a manifold. In Sections 7 and 8 we extend it to any numerable vector bundle using the fact that any numerable vector bundle is a pull-back of the universal bundle, and the universal bundle is filtered by vector bundles over manifolds for which we can apply our construction.

We also prove the uniqueness of the classes $\mathbf{Cl}_i(\xi)$ showing the equivalence of the set of axioms used in the main theorem and the set of axioms defining the Stiefel–Whitney or Chern classes. This also proves that the classes $\mathbf{Cl}_i(\xi)$ are independent of the generic bundle morphism used to define them.

2. Preliminaries

We shall work with real and complex vector bundles. To work with both cases simultaneously we make the following conventions. When $b=1$, let $K_1=\mathbb{Z}_2$ and $\mathbb{F}=\mathbb{R}$ for real vector bundles and when $b=2$, let $K_2=\mathbb{Z}$ and $\mathbb{F}=\mathbb{C}$ for complex vector bundles.

Let $G_n(\mathbb{F}^{n+k})$ be the *Grassmann manifold* of $\mathbb{F}$-subspaces of dimension n in $\mathbb{F}^{n+k}$. One has that $G_n(\mathbb{F}^{n+k})$ is a smooth manifold of dimension bnk [5, 5.1]. As a particular case we have $\mathbb{F}\mathrm{P}^n=G_1(\mathbb{F}^{n+1})$ the n-dimensional $\mathbb{F}$*-projective space* consisting of $\mathbb{F}$-lines in $\mathbb{F}^{n+1}$ through the origin.

Let $\gamma^n(\mathbb{F}^{n+k})=\{(\ell,v)\in G_n(\mathbb{F}^{n+k})\times\mathbb{F}^{n+k}\mid v\in\ell\}$ be the *canonical n-bundle* over $G_n(\mathbb{F}^{n+k})$. We also denote by γ_n^1 the *canonical line bundle* over $\mathbb{F}\mathrm{P}^n$, that is, $\gamma_n^1=\gamma^1(\mathbb{F}^{n+1})$.

For any $n\in\mathbb{N}$ set $\nu(n)=bn-1$ and let $\mathbf{S}^{\nu(n)}=\{\mathbf{x}\in\mathbb{F}^n\mid\|\mathbf{x}\|=1\}$ be the sphere of real dimension $\nu(n)$. Consider $\mathbb{F}\mathrm{P}^n$ as $\mathbf{S}^{\nu(n+1)}/\mathbf{S}^{\nu(1)}$, with $\mathbf{S}^{\nu(1)}$ acting on $\mathbf{S}^{\nu(n+1)}$ by $\lambda\cdot(x_1,\dots,x_{n+1})=(\lambda x_1,\dots,\lambda x_{n+1})$, $\lambda\in\mathbf{S}^{\nu(1)}$ and $(x_1,\dots,x_{n+1})\in\mathbf{S}^{\nu(n+1)}$. We shall denote the elements in $\mathbb{F}\mathrm{P}^n$ by $[\mathbf{x}]$ with $\mathbf{x}\in\mathbf{S}^{\nu(n+1)}$, if $\mathbf{x}=(x_1,\dots,x_{n+1})$ we shall write $[x_1,\dots,x_{n+1}]$. Also consider $\mathbf{S}^{\nu(1)}$ acting on $\mathbf{S}^{\nu(n+1)}\times\mathbb{F}^n$ by $\lambda(\mathbf{x},\mathbf{v})=(\lambda\mathbf{x},\lambda\mathbf{v})$. We define the vector bundle ζ^n over $\mathbb{F}\mathrm{P}^n$ as follows. The total space of ζ^n is given by $E(\zeta^n)=(\mathbf{S}^{\nu(n+1)}\times\mathbb{F}^n)/\mathbf{S}^{\nu(1)}$, we denote the elements in $(\mathbf{S}^{\nu(n+1)}\times\mathbb{F}^n)/\mathbf{S}^{\nu(1)}$ by $[\mathbf{x},\mathbf{v}]$, where $\mathbf{x}\in\mathbf{S}^{\nu(n+1)}$ and $\mathbf{v}\in\mathbb{F}^n$. The projection map is given by $[\mathbf{x},\mathbf{v}]\mapsto[\mathbf{x}]$.

The line bundle γ_n^1 can also be described in an analogous way [12, Proposition 7.7.1]. Its total space is given by $\gamma_n^1=(\mathbf{S}^{\nu(n+1)}\times\mathbb{F})/\mathbf{S}^{\nu(1)}$, where the action of $\mathbf{S}^{\nu(1)}$ is given by $\lambda(\mathbf{x},v)=(\lambda\mathbf{x},\lambda v)$ for every $\lambda\in\mathbf{S}^{\nu(1)}$, $\mathbf{x}\in\mathbf{S}^{\nu(n+1)}$ and $v\in\mathbb{F}$. The projection map is given by $[\mathbf{x},v]\mapsto[\mathbf{x}]$.

Lemma 1. *The bundle ζ^n is isomorphic to the Whitney sum of n copies of the canonical line bundle γ_n^1 over $\mathbb{F}\mathrm{P}^n$.*

Proof. There is a bundle map, i.e. a bundle morphism such that its restriction to each fibre is an isomorphism:

$$\begin{tikzcd}
(\mathbf{S}^{\nu(n+1)} \times \mathbb{F}^n)/\mathbf{S}^{\nu(1)} \arrow[r, "\tilde{\Delta}"] \arrow[d] & \gamma_n^1 \times \cdots \times \gamma_n^1 \arrow[d] \\
\mathbb{F}\mathrm{P}^n \arrow[r, "\Delta"] & \underbrace{\mathbb{F}\mathrm{P}^n \times \cdots \times \mathbb{F}\mathrm{P}^n}_{n}
\end{tikzcd}$$

where Δ is the diagonal map and $\tilde{\Delta}[\mathbf{x}, v_1, \ldots, v_n] = ([\mathbf{x}, v_1], \ldots, [\mathbf{x}, v_n])$. Therefore

$$\Delta^*(\gamma_n^1 \times \cdots \times \gamma_n^1) = \gamma_n^1 \oplus \cdots \oplus \gamma_n^1 \cong \zeta^n. \qquad \square$$

Since $\zeta^1 \cong \gamma_1^1$ we shall call ζ^n the *canonical n-bundle* over $\mathbb{F}\mathrm{P}^n$.

We shall denote by g_n the canonical generator of $H^{bn}(\mathbb{F}\mathrm{P}^n; K_b)$, which is the Kronecker dual of the fundamental class of $\mathbb{F}\mathrm{P}^n$. In the case of $\mathbb{C}\mathrm{P}^n$, this is the class given by its canonical orientation.

3. Axioms for characteristic classes

We consider *characteristic classes* which are cohomology classes in $H^*(B; K_b)$ associated to any $\mathbb{F}$-vector bundle ξ over a base space B which satisfy the following axioms:

Axiom A1. To each vector bundle ξ of rank n there corresponds a sequence of cohomology classes

$$\mathbf{cl}_i(\xi) \in H^{bi}(B; K_b), \quad i = 0, 1, 2, \ldots$$

such that $\mathbf{cl}_0(\xi) = 1$ and $\mathbf{cl}_i(\xi) = 0$ if $i > n$.

Axiom A2. If $f : B' \to B$ is a continuous map then

$$\mathbf{cl}_i\big(f^*(\xi)\big) = f^*\big(\mathbf{cl}_i(\xi)\big).$$

Axiom A3. If ξ and η are vector bundles over B then

$$\mathbf{cl}_i(\xi \oplus \eta) = \sum_{j=0}^{i} \mathbf{cl}_j(\xi) \cup \mathbf{cl}_{i-j}(\eta).$$

Axiom A4. For the canonical line bundle γ_1^1 over $\mathbb{F}\mathrm{P}^1$, $\mathbf{cl}_1(\gamma_1^1) = -g_1 \in H^b(\mathbb{F}\mathrm{P}^1, K_b)$.

The sum $\mathbf{cl}(\xi) = 1 + \mathbf{cl}_1(\xi) + \cdots + \mathbf{cl}_n(\xi) \in H^*(B; K_b)$ is the *total characteristic class*.

In our construction of characteristic classes, instead of proving that they satisfy Axioms A1, A2, A3 and A4, we shall prove that they satisfy the equivalent set of axioms A1, A2, A3′ and A4′ where Axioms A3′ and A4′ are:

Axiom A3′. Let ε^k be the product bundle of rank k. Then

$$\mathbf{Cl}_i\big(\xi \oplus \varepsilon^k\big) = \mathbf{Cl}_i(\xi).$$

Axiom A4′. Let ζ^n be the canonical n-bundle over $\mathbb{F}\mathrm{P}^n$. Then

$$\mathbf{Cl}_n\big(\zeta^n\big) = (-1)^n g_n \in H^{bn}\big(\mathbb{F}\mathrm{P}^n; K_b\big).$$

The equivalence of the sets of Axioms A1, A2, A3 and A4 and A1, A2, A3′ and A4′ is proved in Theorem 15 in Section 8.

We shall denote by $\mathbf{cl}_i(\xi)$ the classes satisfying Axioms A1, A2, A3 and A4, and by $\mathbf{Cl}_i(\xi)$ the classes satisfying Axioms A1, A2, A3′ and A4′.

For real vector bundles the classes $\mathbf{cl}_i(\xi)$ are the *Stiefel–Whitney classes* $w_i(\xi)$. For complex vector bundles the classes $\mathbf{cl}_i(\xi)$ are the *Chern classes* $c_i(\xi)$.

4. Generic vector bundle morphisms

We follow the definition of generic vector bundle morphism given by Macpherson in [9].[2]

Let ζ and ξ be two smooth $\mathbb{F}$-vector bundles over a smooth manifold M. Consider the bundle of morphisms $\pi : \mathrm{Hom}_{\mathbb{F}}(\zeta, \xi) \to M$. A smooth bundle morphism $h : \zeta \to \xi$ is equivalent to a smooth section s_h of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$.

There is a *tautological bundle morphism* over the total space of the bundle $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$. It is a morphism from $\pi^*\zeta$ to $\pi^*\xi$, whose restriction to the fibre over $v \in \mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ is v itself considered as a linear transformation from $\zeta_{\pi(v)} \cong (\pi^*\zeta)_v$ to $\xi_{\pi(v)} \cong (\pi^*\xi)_v$.

A bundle morphism $h : \zeta \to \xi$ induces a partition of the manifold M given by *"singularity" subsets*

$$Z_j(h) = \{x \in M \mid \dim_{\mathbb{F}} \ker h_x = j\} \quad \text{where } 0 \leqslant j \leqslant \operatorname{rank} \zeta.$$

Let $\tau : \pi^*\zeta \to \pi^*\xi$ be the tautological bundle morphism over the bundle of morphisms $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$. Its singularity subset $Z_j(\tau)$ is a subbundle of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ with fibre $Z_j(\tau)_x = \{v \in \mathrm{Hom}_{\mathbb{F}}(\zeta_x, \xi_x) \mid \dim_{\mathbb{F}} \ker v = j\}$.

Lemma 2. *Let $\mathbb{F}(k, n)$ be the vector space of $k \times n$ matrices with entries in $\mathbb{F}$ and let $\mathbb{F}_r(k, n)$ be the subset of $k \times n$ matrices of rank r with $r \leqslant \min\{k, n\}$. Then $\mathbb{F}_r(k, n)$ is a submanifold of $\mathbb{F}(k, n)$ of $\mathbb{F}$-codimension $(k - r)(n - r)$.*

Proof. Consider $\alpha \in \mathbb{F}_r(k, n)$, without loss of generality, interchanging rows and columns, we can assume that the upper left $r \times r$ submatrix of α has non-zero determinant. A chart of $\mathbb{F}(k, n)$ around α is given by the set $U \subset \mathbb{F}(k, n)$ of matrices of the form

$$\begin{pmatrix} A & AB \\ C & CB + D \end{pmatrix}$$

with $A \in \mathbb{F}(r, r)$ and $\det(A) \neq 0$, $B \in \mathbb{F}(r, n - r)$, $C \in \mathbb{F}(k - r, r)$ and $D \in \mathbb{F}(k - r, n - r)$. Adding a multiple of the first column to the second one does not change the rank, so the previous matrix has the same rank as

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

which has rank r if and only if D is the zero matrix. Hence a chart for $\mathbb{F}_r(k, n)$ around α is given by the subset V of U of matrices with $D = 0$. The chart U has dimension kn and V has dimension $r^2 + (k - r)r + r(n - r)$, that is, V has codimension $(k - r)(n - r)$ in U. $\square$

If ζ and ξ have ranks k and n respectively, then by Lemma 2 we have that $Z_j(\tau)_x$ is a submanifold of $\mathrm{Hom}_{\mathbb{F}}(\zeta_x, \xi_x)$ of $\mathbb{F}$-codimension $j(n - k + j)$. Since $Z_j(\tau)_x$ is invariant under the action of the structural group of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ we have that $Z_j(\tau)$ is a submanifold of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ with

$$\operatorname{codim}_{\mathbb{R}} Z_j(\tau) = bj(n - k + j). \tag{1}$$

This was proved by Thom [13, Theorem 2] and it was later generalised by Boardman [14, Theorem (6.1)]. Clearly $Z_{j+1}(\tau)$ belongs to the adherence of $Z_j(\tau)$, thus

$$\bar{Z}_j(\tau) = \bigcup_{l \geqslant j} Z_l(\tau). \tag{2}$$

In fact, the subsets $Z_l(\tau)$ with $l \geqslant j$ give a Whitney stratification of $\bar{Z}_j(\tau)$ (see [15, Chapter I]).

A vector bundle morphism $h : \zeta \to \xi$ is said to be *generic* if the corresponding section s_h of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ is transverse to all the submanifolds $Z_j(\tau)$.

Generic vector bundle morphisms form an open dense subset of the space of all vector bundle morphisms with the Whitney C^∞ topology, this follows for instance from [16, (14.6)].

Proposition 3. *Let ζ and ξ be vector bundles over a manifold M of ranks k and n respectively. If $h : \zeta \to \xi$ is a generic bundle morphism over M, then $Z_j(h)$ is a submanifold of M of* real *codimension $bj(n - k + j)$.*

[2] In [9] Macpherson uses the term generic bundle map.

Proof. Let s_h be the section of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ corresponding to h. On has that $Z_j(h) = s_h^{-1}(Z_j(\tau))$ and since s_h is transverse to $Z_j(\tau)$, $Z_j(h)$ is a submanifold of M of *real* codimension $bj(n-k+j)$. □

Note that also $\bar{Z}_j(h) = \bigcup_{l \geqslant j} Z_l(h)$ and the subsets $Z_l(h)$ with $l \geqslant j$ give a Whitney stratification of $\bar{Z}_j(h)$ (see [15, (1.4)]).

5. The manifold $\tilde{Z}(h)$

Let ξ be a smooth $\mathbb{F}$-vector bundle of rank n over a smooth closed manifold M of dimension m. Also assume that the manifold M is K_b-oriented (see [17, §22] for the definition). Let $h : \varepsilon^k \to \xi$ be a bundle morphism from the product bundle ε^k of rank k to ξ. Define

$$\tilde{Z}(h) = \{(x, L) \in M \times \mathbb{F}\mathrm{P}^{k-1} \mid (x, L) \subset \ker h_x\},$$
$$\tilde{Z}^\circ(h) = \{(x, L) \in \tilde{Z}(h) \mid (x, L) = \ker h_x\}.$$

We have that $\tilde{Z}^\circ(h)$ is an open dense subset of $\tilde{Z}(h)$.

Let τ be the tautological bundle morphism over $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$. We have that

$$\tilde{Z}(\tau) = \{(f, L) \in \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \mid (\pi(f), L) \subset \ker f_{\pi(f)}\}.$$

Proposition 4. [18, Proposition (1.1)] *Let $\hat{\phi} : \tilde{Z}(\tau) \to \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$ be the projection onto the first factor. Then*

(1) $\hat{\phi}(\tilde{Z}(\tau)) = \bar{Z}(\tau)$.
(2) *$\tilde{Z}(\tau)$ is a submanifold of* $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$ *of* real *codimension bn.*

Proof. (1) If $(f, L) \in \tilde{Z}(\tau)$ then $\dim_{\mathbb{F}}(\ker f_{\pi(f)}) \geqslant 1$ and $f \in \bar{Z}_1(\tau)$. On the other hand, if $f \in \bar{Z}_1(\tau)$ then the dimension of the kernel of $f_{\pi(f)}$ is at least 1 and it contains a line L, hence $(f, L) \in \tilde{Z}(\tau)$ and $\hat{\phi}(f, L) = f$.

(2) Let $\pi : \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \to M$ be the bundle of morphisms. Define $\varepsilon' = \hat{\phi}^*\pi^*(\varepsilon^k) = \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \times \mathbb{F}^k$ and $\xi' = \hat{\phi}^*\pi^*(\xi)$. Define the subbundle ε_1 of ε' by

$$\varepsilon_1 = \{(f, L, v) \in \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \times \mathbb{F}^k \mid v \in L\}.$$

Let $\pi' : \mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi') \to \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$ be the bundle of morphisms from ε_1 to ξ'. Define the section $\Psi : \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \to \mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ by

$$\Psi(f, L) = f|_L,$$

where $f|_L$ is the restriction of f to the line L.

We have that $\tilde{Z}(\tau)$ is the set of zeros of the section Ψ. It is easy to see that the section Ψ is transverse to the zero section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$. Hence $\tilde{Z}(\tau)$ is a submanifold of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$. Clearly the zero section has *real* codimension bn in $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ and therefore $\tilde{Z}(\tau)$ has real codimension bn in $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$. □

Proposition 5. *Let $h : \varepsilon^k \to \xi$ be a generic bundle morphism. Then $\tilde{Z}(h)$ is a compact submanifold of $M \times \mathbb{F}\mathrm{P}^{k-1}$ of dimension $m + b(k-n-1)$.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \arrow[r, "\hat{\phi}"] \arrow[d, "\pi \times Id"] & \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \arrow[d, "\pi"'] \\
M \times \mathbb{F}\mathrm{P}^{k-1} \arrow[r, "\phi"] \arrow[u, bend left, "\tilde{s}_h"] & M \arrow[u, bend right, "s_h"']
\end{tikzcd}$$

where $\pi : \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \to M$ is the bundle of morphisms, ϕ and $\hat{\phi}$ are the corresponding projections onto the first factors, s_h is the section of π corresponding to h and the section $\tilde{s}_h$ is given by $\tilde{s}_h = s_h \times Id$.

Let τ be the tautological bundle morphism over $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$. Notice that $\tilde{Z}(h) = \tilde{s}_h^{-1}(\tilde{Z}(\tau))$. Therefore, by Proposition 4(2) it is enough to check that $\tilde{s}_h$ is transverse to $\tilde{Z}(\tau)$.

Let $(x, L) \in \tilde{Z}(h)$, then $\tilde{s}_h(x, L) = (h_x, L) \in \tilde{Z}(\tau)$. By Proposition 4(1) and (2) $h_x \in Z_j(\tau)$ for some $j \geqslant 1$. Hence

$$T_{h_x} Z_j(\tau) \subset d\hat{\phi}_{(h_x,L)}\bigl(T_{(h_x,L)}\tilde{Z}(\tau)\bigr). \tag{3}$$

On the other hand,

$$\begin{aligned} &d(\tilde{s}_h)_{(x,L)}\bigl(T_x M \oplus T_L \mathbb{F}\mathrm{P}^{k-1}\bigr) \oplus T_{(h_x,L)}\tilde{Z}(\tau) \\ &\quad \cong d(s_h)_x(T_x M) \oplus d(Id)_L\bigl(T_L \mathbb{F}\mathrm{P}^{k-1}\bigr) \oplus T_{(h_x,L)}\tilde{Z}(\tau) \\ &\quad \cong d(s_h)_x(T_x M) \oplus T_L \mathbb{F}\mathrm{P}^{k-1} \oplus d\hat{\phi}_{(h_x,L)}\bigl(T_{(h_x,L)}\tilde{Z}(\tau)\bigr) \oplus d\hat{\varphi}_{(h_x,L)}\bigl(T_{(h_x,L)}\tilde{Z}(\tau)\bigr) \end{aligned}$$

where $\hat{\varphi} : \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \to \mathbb{F}\mathrm{P}^{k-1}$ is the projection onto the second factor. By (3) and since $d\hat{\varphi}_{(h_x,L)}(T_{(h_x,L)}\tilde{Z}(\tau)) \subset T_L \mathbb{F}\mathrm{P}^{k-1}$ we have

$$d(s_h)_x(T_x M) \oplus T_L \mathbb{F}\mathrm{P}^{k-1} \oplus T_{h_x} Z_j(\tau) \subset d(\tilde{s}_h)_{(x,L)}\bigl(T_x M \oplus T_L \mathbb{F}\mathrm{P}^{k-1}\bigr) \oplus T_{(h_x,L)}\tilde{Z}(\tau).$$

Since h is generic, s_h is transverse to $Z_j(\tau)$, thus $d(s_h)_x(T_x M) \oplus T_{h_x} Z_j(\tau) \cong T_{h_x} \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$ and

$$T_{h_x} \mathrm{Hom}_{\mathbb{F}}\bigl(\varepsilon^k, \xi\bigr) \oplus T_L \mathbb{F}\mathrm{P}^{k-1} \subset d(\tilde{s}_h)_{(x,L)}\bigl(T_x M \oplus T_L \mathbb{F}\mathrm{P}^{k-1}\bigr) \oplus T_{(h_x,L)}\tilde{Z}(\tau).$$

Since the other inclusion is trivial $\tilde{s}_h$ is transverse to $\tilde{Z}(\tau)$. Hence $\tilde{Z}(h) = \tilde{s}_h^{-1}(\tilde{Z}(\tau))$ is a submanifold of $M \times \mathbb{F}\mathrm{P}^{k-1}$. Since $\tilde{Z}(\tau)$ is closed in $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$, $\tilde{Z}(h)$ is closed in the compact space $M \times \mathbb{F}\mathrm{P}^{k-1}$ and it is therefore compact.

By Proposition 4(2) $\tilde{Z}(h)$ has real codimension bn in $M \times \mathbb{F}\mathrm{P}^{k-1}$ which has real dimension $m + b(k - n - 1)$. $\square$

Proposition 6. *The manifold $\tilde{Z}(h)$ is K_b-oriented. Therefore it has a fundamental class $[\tilde{Z}(h)] \in H_{m+b(k-n-1)}(\tilde{Z}(h); K_b)$.*

Proof. We shall consider two cases.

When $b = 1$, every manifold is $\mathbb{Z}_2$-oriented and since it is compact it has a fundamental class (see for instance [17, Proposition (22.12), Corollary (22.28)]).

Let $b = 2$. If ε^k and ξ are complex vector bundles, then they have a canonical orientation. Hence $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$ also has a canonical orientation. Since M is $\mathbb{Z}$-oriented then $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$ is $\mathbb{Z}$-oriented (as a manifold). Moreover, since the bundle $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ (see the proof of Proposition 4) is a complex bundle, it also has a canonical orientation and since $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$ is a $\mathbb{Z}$-oriented manifold, $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ is also a $\mathbb{Z}$-oriented manifold. The zero section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ is a $\mathbb{Z}$-oriented manifold being diffeomorphic to $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$. Since $\tilde{Z}(\tau)$ is the inverse image of the zero section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ under the section Ψ defined in the proof of Proposition 4(2), then it is also $\mathbb{Z}$-oriented. On the other hand, if the bundle morphism $h : \varepsilon^k \to \xi$ is generic, by the proof of Proposition 5, the section $\tilde{s}_h$ is transverse to $\tilde{Z}(\tau)$. Finally, since $\tilde{Z}(h) = \tilde{s}_h^{-1}(\tilde{Z}(\tau))$, we have that $\tilde{Z}(h)$ is a $\mathbb{Z}$-oriented manifold. $\square$

Proposition 7. *Let $\phi : \tilde{Z}(h) \to M$ be the projection onto the first factor. Then ϕ is proper and maps $\tilde{Z}^\circ(h)$ diffeomorphically onto $Z_1(h)$.*

Proof. The image of ϕ is $\bar{Z}_1(h) = \bigcup_{l \geqslant 1} Z_l(h)$. Since $M \times \mathbb{F}\mathrm{P}^{k-1}$ and M are compact $\phi : M \times \mathbb{F}\mathrm{P}^{k-1} \to M$ is proper. The subspaces $\tilde{Z}(h)$ and $\bar{Z}(h)$ are closed in $M \times \mathbb{F}\mathrm{P}^{k-1}$ and M respectively, hence the restriction of ϕ to $\tilde{Z}(h)$ is proper. We have that

$$\phi^{-1}\bigl(Z_1(h)\bigr) = \bigl\{(x, L) \in \tilde{Z}(h) \mid \ker h_x = L\bigr\} = \tilde{Z}^\circ(h).$$

For every $x \in Z_1(h)$ we have that $\dim \ker h_x = 1$, thus x has only one preimage in $\tilde{Z}^\circ(h)$. Since $\tilde{Z}^\circ(h)$ is open in $\tilde{Z}(h)$ it is a submanifold, so ϕ restricted to $\tilde{Z}^\circ(h)$ is a diffeomorphism. $\square$

Lemma 8. *Let M and M' be two closed K_b-oriented differentiable manifolds and let Z be a closed K_b-oriented submanifold of M. Let $f : M' \to M$ be a differentiable map transverse to Z and set $Z' = f^{-1}(Z)$. Denote by i and j the inclusions of Z and Z' in M and M' respectively. If $[Z]$ and $[Z']$ are the corresponding fundamental classes of Z and Z', then*

$$j_*([Z']) = D_{M'} \circ f^* \circ D_M^{-1} \circ i_*([Z]),$$

where D_M and $D_{M'}$ denote the Poincaré duality isomorphisms in M and in M' respectively.

Proof. Since f is transverse to Z, then Z' is a K_b-oriented submanifold of M' with the preimage orientation (see for instance [19, p. 100]).

Set $\dim M = m$, $\dim M' = m'$, $\dim Z = r$, $\dim Z' = r'$ and since Z and Z' have the same codimension let $q = m - r = m' - r'$.

Let ν be the normal bundle of Z in M, it is oriented so that $\nu \oplus TZ$ is orientation preserving isomorphic to the restriction of TM to Z. Let ν' be the normal bundle of Z' in M', then we have that $\nu' = f^*\nu$ and in this way ν' gets its orientation from the orientation of ν.

Let $E(\nu)$ be the total space of ν and let $E(\nu)_0$ be the set of all non-zero elements of $E(\nu)$. Analogously define $E(\nu')$ and $E(\nu')_0$ for ν'. We have the following canonical isomorphisms of cohomology rings [5, Corollary 11.2]

$$H^*(E(\nu), E(\nu)_0; K_b) \cong H^*(M, M - Z; K_b),$$
$$H^*(E(\nu'), E(\nu')_0; K_b) \cong H^*(M', M' - Z'; K_b).$$

Under these isomorphisms the Thom classes of ν and ν' correspond to canonical classes

$$u_\nu \in H^q(M, M - Z; K_b), \qquad u_{\nu'} \in H^q(M', M' - Z'; K_b).$$

Consider the following commutative diagram

$$\begin{tikzcd}
H^q(M, M - Z; K_b) \arrow[r, "\iota"] \arrow[d, "f^*"'] & H^q(M; K_b) \arrow[r, "D_M"] \arrow[d, "f^*"'] & H_r(M; K_b) & H_r(Z; K_b) \arrow[l, "i_*"'] \\
H^q(M', M' - Z'; K_b) \arrow[r, "\iota'"] & H^q(M'; K_b) \arrow[r, "D_{M'}"] & H_{r'}(M'; K_b) & H_{r'}(Z'; K_b) \arrow[l, "j_*"']
\end{tikzcd} \tag{4}$$

We have that (see [5, Problem 11.C])[3]

$$\iota(u_\nu) = D_M^{-1} i_*([Z]),$$
$$\iota'(u_{\nu'}) = D_{M'}^{-1} j_*([Z']).$$

The lemma follows from the commutativity of (4) and the fact that $u_{\nu'} = f^*(u_\nu)$. $\square$

Proposition 9. *Let M, M' and P be K_b-oriented closed manifolds. Consider the following commutative diagram*

$$\begin{tikzcd}
Q \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P \arrow[d, "g"] \\
M' \arrow[r, "f"'] & M
\end{tikzcd}$$

where f and g are differentiable maps, Q is the fibred product given by

$$Q = \{(y, p) \in M' \times P \mid f(y) = g(p)\},$$

and π_1 and π_2 are the projections. Suppose that f and g are transverse and let $[P]$ and $[Q]$ be the fundamental classes of P and Q respectively. Then

$$(-1)^{(m-r)m} \pi_{1*}([Q]) = D_{M'} \circ f^* \circ D_M^{-1} \circ g_*([P]),$$

[3] We are using the sign conventions of [5] but we are orienting ν in a different way.

where D_M and $D_{M'}$ denote the Poincaré duality isomorphisms in M and M' respectively and $m = \dim M$ and $r = \dim P$.

Proof. Let $i : \Delta \to M \times M$ be the inclusion of the diagonal. We have that f is transverse to g if and only if $(f \times g)$ is transverse to Δ where $f \times g : M' \times P \to M \times M$. Therefore $Q = (f \times g)^{-1}(\Delta)$ is a K_b-oriented submanifold of $M' \times P$ of dimension $r' = r + m' - m$, with $m' = \dim M'$. Let $[\Delta]$ be the fundamental class of Δ and set $u_\Delta = D^{-1}_{M\times M}(i_*([\Delta]))$. If $j : Q \to M' \times P$ is the inclusion then by Lemma 8

$$\begin{aligned} j_*\big([Q]\big) &= D_{M'\times P} \circ (f \times g)^*(u_\Delta) \\ &= (f \times g)^*(u_\Delta) \cap [M' \times P], \end{aligned}$$

where $[M' \times P]$ is the fundamental class of $M' \times P$. Let $\bar{\pi}_1 : M' \times P \to M'$ be the projection onto the first factor, since $\pi_1 = \bar{\pi}_1 \circ j$ we have (see [20, Proposition 13.61(vi)])

$$\begin{aligned} &\bar{\pi}_{1*}\big(j_*\big([Q]\big)\big) = \pi_{1*}\big((f \times g)^*(u_\Delta) \cap [M' \times P]\big), \\ &\pi_{1*}\big([Q]\big) = \big((f \times g)^*(u_\Delta)/[P]\big) \cap [M'] \end{aligned} \tag{5}$$

where $[M']$ and $[P]$ are the fundamental classes of M' and P respectively. A well-known property of the slant product (see [17, (29.23)]) gives the following commutative diagram

$$\begin{tikzcd} H_r(M; K_b) \arrow[r, "u_\Delta/"] & H^{m-r}(M; K_b) \arrow[d, "f^*"] & \\ H_r(P; K_b) \arrow[u, "g_*"] \arrow[r, "(g\times f)^*(u_\Delta)/"'] & H^{m-r}(M'; K_b) \arrow[r, "\bigcap[M']"'] & H_{r'}(M'; K_b) \end{tikzcd}$$

Applying the two different compositions to $(-1)^{(m-r)m}[P] \in H_r(P; K_b)$ and using (5) we have

$$\begin{aligned} &(-1)^{(m-r)m}\big((f \times g)^*(u_\Delta)/[P]\big) \cap [M'] = f^*\big((-1)^{(m-r)m} u_\Delta/g_*\big([P]\big)\big) \cap [M'], \\ &(-1)^{(m-r)m}\pi_{1*}\big([Q]\big) = D_{M'} \circ f^*\big((-1)^{(m-r)m} u_\Delta/g_*\big([P]\big)\big). \end{aligned}$$

This proves the proposition since the homomorphism $(-1)^{(m-r)m} u_\Delta/$ is the inverse of the Poincaré duality isomorphism D_M [17, Theorem (30.6)]. □

Remark 10. Lemma 8 and Proposition 9 are generalisations of a result proved by Gottlieb [21, Proposition 2]. If in Proposition 9 we consider the case when P is a submanifold of M and g is the inclusion then we are in the situation of Lemma 8. In this case the sign reflects the difference between the orientations given to Q as the preimage of P under f or as the preimage of Δ under $f \times g$ followed by the projection π_1.

Given a smooth map $f : M' \to M$ the homomorphism $f_!$ defined by $f_! = D_{M'} \circ f^* \circ D_M^{-1}$ is called the Hopf-Umkehrhomomorphismus. Proposition 9 is equivalent to the equality $(-1)^{(m-r)m}\pi_{1*}\pi_{2!} = f_! g_*$, since clearly $[Q] = \pi_{2!}([P])$. An analogous formula was proved in Cobordism Theory by Quillen [22].

6. Definition of the classes $\mathbf{Cl}_i(\xi)$

Theorem 11. *Let ξ be a smooth $\mathbb{F}$-vector bundle of rank n over a smooth closed K_b-oriented manifold M of dimension m. Let $h : \varepsilon^{n-i+1} \to \xi$ be a generic bundle morphism from the product bundle ε^{n-i+1} of rank $n - i + 1$ to ξ. Then the classes*

$$\mathbf{Cl}_i(\xi) = \hat{\phi}\big(\big[\tilde{Z}(h)\big]\big) \in H^{bi}(M; K_b)$$

satisfy Axioms A1, A2, A3$'$ *and* A4$'$, *where $[\tilde{Z}(h)]$ is the fundamental class of $\tilde{Z}(h)$, and $\hat{\phi}$ is the composition of the Poincaré duality isomorphism with the homomorphism induced in homology by the projection onto the first factor $\phi : \tilde{Z}(h) \to M$*

$$\begin{tikzcd} H_{m-bi}(\tilde{Z}(h);K_b) \arrow[r, "\phi_*"] \arrow[dr, "\hat{\phi}"'] & H_{m-bi}(M;K_b) \arrow[d, leftrightarrow, "D"] \\ & H^{bi}(M;K_b) \end{tikzcd}$$

Proof. We shall show that the classes $\mathbf{Cl}_i(\xi)$ satisfy Axioms A1, A2, A3′ and A4′.

Axiom A1. By Proposition 5, $\tilde{Z}(h)$ has dimension $m-bi$. Therefore $\mathbf{Cl}_i(\xi)\in H^{bi}(M;K_b)$.

If $i=0$ then $\phi(\tilde{Z}(h))=M$. Hence $\bar{Z}_1(h)=M$ and by Proposition 7 any $x\in Z_1(h)$ is a regular value of ϕ with only one preimage. Therefore ϕ has degree 1 and $\mathbf{Cl}_0(\xi)=1\in H^0(M;K_b)$.

The constructions does not make sense for $i>n$ so we set $\mathbf{Cl}_i(\xi)=0\in H^{bi}(M;K_b)$ for $i>n$.

Axiom A2. Let $f:M'\to M$ be a differentiable map and consider the pull-back diagram

$$\begin{tikzcd} f^*\xi \arrow[r, "\bar{f}"] \arrow[d, "p'"'] & \xi \arrow[d, "p"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

Let $h:\varepsilon_M^{n-i+1}\to\xi$ be a generic bundle morphism. Recall that by Proposition 3 the singularity subsets $Z_j(h)$ of h are submanifolds of M. Without loss of generality we can assume that f is transverse to all the $Z_j(h)$ (if not find a map homotopic to f which is [16, (14.7)]). Consider the bundle of morphisms $\pi:\mathrm{Hom}_{\mathbb{F}}(\varepsilon_M^{n-i+1},\xi)\to M$, since $f^*\,\mathrm{Hom}_{\mathbb{F}}(\varepsilon_M^{n-i+1},\xi)\cong\mathrm{Hom}_{\mathbb{F}}(f^*\varepsilon_M^{n-i+1},f^*\xi)$ and $f^*\varepsilon_M^{n-i+1}\cong\varepsilon_{M'}^{n-i+1}$ we have the following pull-back diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{F}}(\varepsilon_{M'}^{n-i+1},f^*\xi) \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"] & \mathrm{Hom}_{\mathbb{F}}(\varepsilon_M^{n-i+1},\xi) \arrow[d, "\pi"'] \\ M' \arrow[r, "f"] \arrow[u, bend left, "s_g"] & M \arrow[u, bend right, "s_h"'] \end{tikzcd} \tag{6}$$

where s_h is the section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_M^{n-i+1},\xi)$ corresponding to h.

Let τ and τ' be the tautological bundle morphisms over $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_M^{n-i+1},\xi)$ and $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_{M'}^{n-i+1},f^*\xi)$ respectively. To simplify notation set $S_j=Z_j(\tau)$ and $S'_j=Z_j(\tau')$. We have that $S'_j=\tilde{f}^{-1}(S_j)$.

Define a section s_g of $\pi':\mathrm{Hom}_{\mathbb{F}}(\varepsilon_{M'}^{n-i+1},f^*\xi)\to M'$ by

$$s_g(y)=\bigl(y,\tilde{f}_y^{-1}\bigl(s_h\bigl(f(y)\bigr)\bigr)\bigr)$$

which is well defined since $\tilde{f}$ is isomorphism restricted to the fibres.

We need to check that s_g is transverse to every S'_j. Since h is generic, we have that s_h is transverse to S_j for every j. On the other hand, $Z_j(h)=s_h^{-1}(S_j)$ and we choose f such that f is transverse to $Z_j(h)$ for every j. This is equivalent to $s_h\circ f$ being transverse to S_j for every j. Since diagram (6) commutes we have that $\tilde{f}\circ s_g$ is transverse to S_j for every j, which is equivalent to s_g being transverse to S'_j for every j, since $S'_j=\tilde{f}^{-1}(S_j)$ and clearly $\tilde{f}$ is transverse to S_j. From (6) we also have that $Z_j(g)=f^{-1}(Z_j(h))$ for every j so $\bar{Z}_j(g)=f^{-1}(\bar{Z}_j(h))$.

Let $\phi:\tilde{Z}(h)\to M$ and $\phi':\tilde{Z}(g)\to M'$ be respectively the manifolds and maps corresponding to the generic bundle morphisms h and g given by Proposition 7. Since the image of ϕ is $\bar{Z}_1(h)$ and f is transverse to all the $Z_j(h)$ we have that f is transverse to ϕ. Moreover, the transverse intersection of f and ϕ is diffeomorphic to $\tilde{Z}(g)$ since

$$\begin{aligned} M'\pitchfork\tilde{Z}(h) &= \bigl\{(y,x,L)\in M'\times\tilde{Z}(h)\mid f(y)=\phi(x,L),\ (x,L)\subset\ker_{h_x}\bigr\}\\ &= \bigl\{\bigl(y,f(y),L\bigr)\in M'\times\tilde{Z}(h)\mid\bigl(f(y),L\bigr)\subset\ker_{h_{f(y)}}\bigr\}\\ &\cong \bigl\{(y,L)\in M'\times\mathbb{F}\mathrm{P}^{n-i}\mid(y,L)\subset\ker_{g_y}\bigr\}\\ &= \tilde{Z}(g). \end{aligned}$$

Hence we have the following commutative diagram

$$\begin{tikzcd} \tilde{Z}(g) \arrow[r] \arrow[d, "\phi'"'] & \tilde{Z}(h) \arrow[d, "\phi"] \\ M' \arrow[r, "f"'] & M \end{tikzcd}$$

which satisfies the hypothesis of Proposition 9 and therefore

$$(-1)^{bim}\phi'_*\big(\big[\tilde{Z}(g)\big]\big) = D_{M'} \circ f^* \circ D_M^{-1} \circ \phi_*\big(\big[\tilde{Z}(h)\big]\big).$$

For the real case ($b = 1$), we use $\mathbb{Z}_2$-coefficients so the sign is not important. For the complex case, since $b = 2$ the sign is always positive. Then one has that $f^*: H^*(M; K_b) \to H^*(M'; K_b)$ maps the Poincaré dual of the class $\phi_*([\tilde{Z}(h)])$ to the Poincaré dual of the class $\phi'_*[\tilde{Z}(g)]$. Thus $\mathbf{Cl}_i(f^*\xi) = f^*(\mathbf{Cl}_i(\xi))$ and Axiom A2 is satisfied.

Axiom A3′. Let $h: \varepsilon^{n-i+1} \to \xi$ be a generic bundle morphism. Consider the vector bundle morphism $h \oplus \mathrm{id}_{\varepsilon^k}: \varepsilon^{n-i+1} \oplus \varepsilon^k \to \xi \oplus \varepsilon^k$. Note that

$$Z_j(h \oplus \mathrm{id}_{\varepsilon^k}) = Z_j(h) \quad \text{for all } j. \tag{7}$$

We need to check that $h \oplus \mathrm{id}_{\varepsilon^k}$ is generic. Consider the bundle morphism

$$\begin{tikzcd}[column sep=huge] \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi) \arrow[rr, "\hat{f}"] \arrow[dr] & & \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k) \arrow[dl] \\ & M \arrow[ul, bend left, "s_h"] \arrow[ur, bend right, "s_{h\oplus \mathrm{id}_{\varepsilon^k}}"'] & \end{tikzcd}$$

given by $\hat{f}(v) = v \oplus \mathrm{id}_{\varepsilon^k}$. Let τ and τ'' be the tautological bundle morphisms over $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)$ and $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k)$ respectively. Again to simplify notation set $S_j = Z_j(\tau)$ and $S''_j = Z_j(\tau'')$. We have that $S_j = \hat{f}^{-1}(S''_j)$. As before let s_h be the section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)$ corresponding to h. Since h is generic s_h is transverse to all the S_j. Let $s_{h\oplus \mathrm{id}_{\varepsilon^k}}$ be the section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k)$ corresponding to $h \oplus \mathrm{id}_{\varepsilon^k}$. We have that $s_{h\oplus \mathrm{id}_{\varepsilon^k}} = \hat{f} \circ s_h$, and to prove that $s_{h\oplus \mathrm{id}_{\varepsilon^k}}$ is transverse to all the S''_j it is enough to show that $\hat{f}$ is transverse to all the S''_j. By (1) we have the following equalities

$$\begin{aligned}
&\dim \mathrm{Hom}_{\mathbb{F}}\big(\varepsilon^{n-i+1}, \xi\big) = b^2(n-i+1)n + m, \\
&\dim S_j = b^2(n-i+1)n + m - bj(n-k+j), \\
&\dim \mathrm{Hom}_{\mathbb{F}}\big(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k\big) = b^2(n-i+1+k)(n+k) + m, \\
&\dim S''_j = b^2(n-i+1+k)(n+k) + m - bj(n-k+j).
\end{aligned}$$

Let $v \in S_j$ and $w = \hat{f}(v)$, we need to prove that

$$\dim\big\{d\hat{f}_v\big(T_v \mathrm{Hom}_{\mathbb{F}}\big(\varepsilon^{n-i+1}, \xi\big)\big) \oplus T_w S''_j\big\} = \dim\big\{T_w \mathrm{Hom}_{\mathbb{F}}\big(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k\big)\big\}.$$

The vectors in $T_v \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)$ which are sent to $T_w S''_j$ by $d\hat{f}_v$ are precisely the vectors in $T_v S_j$ and $d\hat{f}_v$ restricted to $T_v S_j$ is an isomorphism, so $d\hat{f}_v(T_v \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)) \cap T_w S''_j \cong T_v S_j$. Then

$$\begin{aligned}
\dim\big\{d\hat{f}_v\big(T_v \mathrm{Hom}_{\mathbb{F}}\big(\varepsilon^{n-i+1}, \xi\big)\big) \oplus T_w S''_j\big\} &= \dim d\hat{f}_v\big(T_v \mathrm{Hom}_{\mathbb{F}}\big(\varepsilon^{n-i+1}, \xi\big)\big) + \dim T_w S''_j - \dim T_v S_j \\
&= b^2(n-i+1+k)(n+k) + m \\
&= \dim \mathrm{Hom}_{\mathbb{F}}\big(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k\big).
\end{aligned}$$

Hence $h \oplus \mathrm{id}_{\varepsilon^k}$ is generic.

By Proposition 5, $\tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k})$ is a submanifold of $M \times \mathbb{F}\mathrm{P}^{n-i+k}$ of dimension $m - bi$. Let $\check{\phi} : \tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k}) \to M$ be the projection onto the first factor. Let $\Phi : M \times \mathbb{F}\mathrm{P}^{n-i} \to M \times \mathbb{F}\mathrm{P}^{n-i+k}$ be the inclusion given by

$$\Phi\big(\big(x, [x_1, \dots, x_{n-i+1}]\big)\big) = \big(x, [x_1, \dots, x_{n-i+1}, \underbrace{0, \dots, 0}_{k}]\big),$$

where $[x_1, \dots, x_{n-i}] \in \mathbb{F}\mathrm{P}^{n-i}$ is given in homogeneous coordinates. We have that Φ maps diffeomorphically $\tilde{Z}(h)$ onto $\tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k})$. By (7) we have that $\bar{Z}_1(h \oplus \mathrm{id}_{\varepsilon^k}) = \bar{Z}_1(h)$. Hence the following diagram commutes

$$\begin{tikzcd}
\tilde{Z}(h) \arrow[d, "\Phi"'] \arrow[dr, "\phi"] & \\
 & M \\
\tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k}) \arrow[ur, "\check{\phi}"'] &
\end{tikzcd}$$

Thus $\phi_*([\tilde{Z}(h)]) = \check{\phi}_*([\tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k})])$ and therefore $\mathbf{Cl}_i(\xi \oplus \varepsilon^k) = \mathbf{Cl}_i(\xi)$. Hence Axiom A3$'$ is satisfied.

Axiom A4$'$. We need to check that if ζ^n is the canonical n-bundle over $\mathbb{F}\mathrm{P}^n$ then we have that $\mathbf{Cl}_n(\zeta^n) = (-1)^n g_n \in H^{bn}(\mathbb{F}\mathrm{P}^n; K_b)$.

Define the vector bundle morphism $h : \varepsilon^1 \to \zeta^n$ by

$$h\big(\big([x_1, \dots, x_{n+1}], t\big)\big) = [x_1, \dots, x_{n+1}, tx_1, \dots, tx_n].$$

We have that $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n) \cong \zeta^n$ and its section s_h corresponding to h is given by

$$s_h\big([x_1, \dots, x_{n+1}]\big) = [x_1, \dots, x_{n+1}, x_1, \dots, x_n].$$

Let τ be the tautological vector bundle morphism over $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$, the only singularity subset is $S_1 = Z_1(\tau)$ and it is equal to the zero section. Let $\mathbf{x_0} = [0, \dots, 0, 1] \in \mathbb{F}\mathrm{P}^n$ and let $R = s_h(\mathbb{F}\mathrm{P}^n) \subset \mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$. We have that R intersects S_1 only in the point $[\mathbf{x_0}, \mathbf{0}]$. We need to prove that R and S_1 are transverse, but before doing it let us make the following convention to simplify notation. Let $\mathbf{x} = (x_1, \dots, x_{n+1}) \in \mathbb{F}^{n+1}$, we set $\tilde{\mathbf{x}} = (x_1, \dots, x_n) \in \mathbb{F}^n$ and we shall write $\mathbf{x} = (\tilde{\mathbf{x}}, x_{n+1})$. The tangent space of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$ at $[\mathbf{x}, \mathbf{v}]$ is given by

$$T_{[\mathbf{x},\mathbf{v}]} \mathrm{Hom}_{\mathbb{F}}\big(\varepsilon^1, \zeta^n\big) = \big\{\langle \mathbf{y}, \mathbf{w}\rangle \mid (\mathbf{y}, \mathbf{w}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n,\ \mathbf{x} \cdot \mathbf{y} = 0\big\}$$

where $\langle \mathbf{y}, \mathbf{w}\rangle$ is the orbit of $(\mathbf{y}, \mathbf{w}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n$ under the action of $\mathbf{S}^{\nu(1)}$ given by $\lambda(\mathbf{y}, \mathbf{w}) = (\lambda\mathbf{y}, \lambda\mathbf{w})$. Hence

$$T_{[\mathbf{x_0},\mathbf{0}]} \mathrm{Hom}_{\mathbb{F}}\big(\varepsilon^1, \zeta^n\big) = \big\{\langle \mathbf{y}, \mathbf{w}\rangle \mid (\mathbf{y}, \mathbf{w}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n,\ \mathbf{y} = (\tilde{\mathbf{y}}, 0)\big\},$$

since the condition $\mathbf{x}_0 \cdot \mathbf{y} = 0$ is equivalent to $\mathbf{y}$ having the last coordinate equal to zero. We also have

$$T_{[\mathbf{x_0},\mathbf{0}]} S_1 = \big\{\langle \mathbf{y}, \mathbf{0}\rangle \mid (\mathbf{y}, \mathbf{0}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n,\ \mathbf{y} = (\tilde{\mathbf{y}}, 0)\big\},$$
$$T_{[\mathbf{x_0},\mathbf{0}]} R = \big\{\langle \mathbf{y}, \mathbf{w}\rangle \mid (\mathbf{y}, \mathbf{w}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n,\ \mathbf{y} = (\mathbf{w}, 0)\big\}.$$

We can write $\langle (\tilde{\mathbf{y}}, 0), \mathbf{w}\rangle \in T_{[\mathbf{x_0},\mathbf{0}]} \mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$ as

$$\big\langle (\tilde{\mathbf{y}}, 0), \mathbf{w}\big\rangle = \big\langle (\mathbf{w}, 0), \mathbf{w}\big\rangle + \big\langle (\tilde{\mathbf{y}} - \mathbf{w}, 0), \mathbf{0}\big\rangle.$$

We have that $\langle (\mathbf{w}, 0), \mathbf{w}\rangle \in T_{[\mathbf{x_0},\mathbf{0}]} R$ and $\langle (\tilde{\mathbf{y}} - \mathbf{w}, 0), \mathbf{0}\rangle \in T_{[\mathbf{x_0},\mathbf{0}]} S_1$. Therefore h is generic and $Z_1(h) = \{\mathbf{x_0}\}$ is a submanifold of dimension zero.

When $\mathbb{F} = \mathbb{C}$ we have to take into account orientations. Since $\mathbb{F}\mathrm{P}^0$ is a point we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n) \times \mathbb{F}\mathrm{P}^0 \arrow[r, "\hat{\phi}", "\cong"'] \arrow[d, "\pi \times \mathrm{Id}"] & \mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n) \arrow[d, "\pi"'] \\
\mathbb{F}\mathrm{P}^n \times \mathbb{F}\mathrm{P}^0 \arrow[r, "\phi", "\cong"'] \arrow[u, bend left, "\tilde{s}_h"] & M \arrow[u, bend right, "s_h"']
\end{tikzcd}$$

Hence we can identify $\tilde{Z}(\tau)$ with S_1 and $\tilde{Z}(h)$ with $Z_1(h)$. We have to take into account the orientations of $\mathbb{C}\mathrm{P}^n$, S_1 and $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$ in order to determine the orientation of $Z_1(h)$ and therefore the sign of $[\tilde{Z}(h)] = [Z_1(h)]$. Let $\{e_1, \dots, e_{n+1}\}$ and $\{f_1, \dots, f_n\}$ be the canonical basis of $\mathbb{C}^{n+1}$ and of $\mathbb{C}^n$ respectively.

The image of the canonical basis of $T_{\mathbf{x_0}}\mathbb{C}\mathrm{P}^n$ under $d(s_h)_{x_0}$ is the basis of $T_{[\mathbf{x_0},\mathbf{0}]}R$ given by $\{\langle e_1, f_1\rangle, \dots, \langle e_n, f_n\rangle\}$. On the other hand, the canonical basis for $T_{[\mathbf{x_0},\mathbf{0}]}S_1$ and for $T_{[\mathbf{x_0},\mathbf{0}]}\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$ are $\{\langle e_1, 0\rangle, \dots, \langle e_n, 0\rangle\}$ and $\{\langle e_1, 0\rangle, \dots, \langle e_n, 0\rangle, \langle 0, f_1\rangle, \dots, \langle 0, f_n\rangle\}$ respectively. Since the s_h is transverse to S_1 a basis for $T_{[\mathbf{x_0},\mathbf{0}]}R \oplus T_{[\mathbf{x_0},\mathbf{0}]}S_1$ is given by

$$\bigl\{\langle e_1, f_1\rangle, \dots, \langle e_n, f_n\rangle, \langle e_1, 0\rangle, \dots, \langle e_n, 0\rangle\bigr\}.$$

Clearly

$$\operatorname{sign}\bigl\{\langle e_1, 0\rangle, \dots, \langle e_n, 0\rangle, \langle 0, f_1\rangle, \dots, \langle 0, f_n\rangle\bigr\} = (-1)^{n^2} \operatorname{sign}\bigl\{\langle e_1, f_1\rangle, \dots, \langle e_n, f_n\rangle, \langle e_1, 0\rangle, \dots, \langle e_n, 0\rangle\bigr\}.$$

Therefore $[\tilde{Z}(h)] = (-1)^{n^2}\mathbf{x_0} \in H_0(\{\mathbf{x_0}\}; K_b)$ and $\mathbf{Cl}_n(\zeta^n) = \phi!([\tilde{Z}(h)]) = (-1)^{n^2} g_n \in H^{bn}(\mathbb{F}\mathrm{P}^n; K_b)$. Hence Axiom A4′ is satisfied. □

7. Generalisation

So far the classes $\mathbf{Cl}_i(\xi)$ are defined only for smooth vector bundles over manifolds.

They can be extended to any numerable vector bundle using the fact that any numerable vector bundle is a pull-back of the universal bundle, and the universal bundle is filtered by vector bundles over manifolds for which we can apply our construction.

Let $\gamma^n(\mathbb{F}^{n+l})$ be the canonical bundle over the Grassmann manifold $G_n(\mathbb{F}^{n+l})$. The inclusions $\mathbb{F}^{n+l} \subset \mathbb{F}^{n+l+1} \subset \cdots$ give inclusions $G_n(\mathbb{F}^{n+l}) \subset G_n(\mathbb{F}^{n+l+1}) \subset \cdots$ and $\gamma^n(\mathbb{F}^{n+l}) \subset \gamma^n(\mathbb{F}^{n+l+1}) \subset \cdots$. We set $G_n = G_n(\mathbb{F}^\infty) = \bigcup_l G_n(\mathbb{F}^{n+l})$ and $\gamma^n = \bigcup_l \gamma^n(\mathbb{F}^{n+l})$ with the direct limit topologies. Let $\iota_l : G_n(\mathbb{F}^{n+l}) \to G_n$ be the inclusions.

One has the following isomorphism

$$\begin{aligned} H^{bi}(G_n; K_b) &\xrightarrow{\lambda} \varprojlim H^{bi}\bigl(G_n\bigl(\mathbb{F}^{n+l}\bigr); K_b\bigr) \\ \omega &\mapsto \bigl((\iota_0)_*(\omega), (\iota_1)_*(\omega), \dots, (\iota_k)_*(\omega), \dots\bigr). \end{aligned}$$

The case $\mathbb{F} = \mathbb{R}$ follows from [23, Proposition 3F.5]. The case $\mathbb{F} = \mathbb{C}$ follows from [20, Proposition 7.66 & Theorem 7.75] since by [12, Theorem 20.3.2] and (8), the groups $H^{bi}(G_n(\mathbb{C}^{n+l}); \mathbb{Z})$ satisfy the Mittag-Leffler condition [20, Definition 7.74].

Let $\gamma^n(\mathbb{F}^{n+l})$ be the canonical bundle over $G_n(\mathbb{F}^{n+l})$ and let γ^n be the universal bundle over G_n. We have the following pull-back diagram

$$\begin{tikzcd} \gamma^n(\mathbb{F}^{n+l}) \arrow[r] \arrow[d] & \gamma^n(\mathbb{F}^{n+l+1}) \arrow[d] \\ G_n(\mathbb{F}^{n+l}) \arrow[r, hook, "\iota"] & G_n(\mathbb{F}^{n+l+1}) \end{tikzcd} \tag{8}$$

then by Axiom A2 $\iota^*(\mathbf{Cl}_i(\gamma^n(\mathbb{F}^{n+l+1}))) = \mathbf{Cl}_i(\gamma^n(\mathbb{F}^{n+l}))$. Hence we have the element $(\mathbf{Cl}_i(\gamma^n(\mathbb{F}^n)), \dots, \mathbf{Cl}_i(\gamma^n(\mathbb{F}^{n+l})), \dots)$ in $\varprojlim H^i(G_n(\mathbb{F}^{n+l}); K_b)$ and we can define

$$\mathbf{Cl}_i\bigl(\gamma^n\bigr) = \lambda^{-1}\bigl(\bigl(\mathbf{Cl}_i\bigl(\gamma^n\bigl(\mathbb{F}^n\bigr)\bigr), \dots, \mathbf{Cl}_i\bigl(\gamma^n\bigl(\mathbb{F}^{n+l}\bigr)\bigr), \dots\bigr)\bigr) \in H^{bi}(G_n; K_b). \tag{9}$$

Therefore we have

$$\iota_l^*\bigl(\mathbf{Cl}_i\bigl(\gamma^n\bigr)\bigr) = \mathbf{Cl}_i\bigl(\gamma^n\bigl(\mathbb{F}^{n+l}\bigr)\bigr) \quad \text{for all } l \geqslant 0. \tag{10}$$

Note that for $\mathbb{F} = \mathbb{C}$, since $G_n(\mathbb{C}^{n+l})$ is a complex manifold, it has a natural orientation and therefore the classes $\mathbf{Cl}_i(\gamma^n(\mathbb{F}^{n+l}))$ are well-defined with $\mathbb{Z}$ coefficients.

Proposition 12. *Let ξ be an $\mathbb{F}$-vector bundle of rank n over a compact K_b-oriented smooth manifold M. Let $\psi_\xi : M \to G_n$ be the classifying map of ξ. Then*

$$\mathbf{Cl}_i(\xi) = \psi_\xi^*\bigl(\mathbf{Cl}_i\bigl(\gamma^n\bigr)\bigr).$$

Proof. For l sufficiently large there exist a map $\varrho_l : M \to G_n(\mathbb{F}^{n+l})$ such that $\xi = \varrho_l^*(\gamma^n(\mathbb{F}^{n+l}))$ [5, Lemma 5.3]. By Axiom A2, $\mathbf{Cl}_i(\xi) = \varrho_l^*(\mathbf{Cl}_i(\gamma^n(\mathbb{F}^{n+l})))$. On the other hand, since $\xi = (\iota_l \circ \varrho_l)^*(\gamma^n)$ where $\iota_l : G_n(\mathbb{F}^{n+l}) \to G_n$ is the inclusion, we have that ψ_ξ and $\iota_l \circ \varrho_l$ are homotopic [5, Theorem 5.7]. Hence using (10) we have

$$\begin{aligned}\psi_\xi^*\big(\mathbf{Cl}_i\big(\gamma^n\big)\big) &= (\iota_l \circ \varrho_l)^*\big(\mathbf{Cl}_i\big(\gamma^n\big)\big)\\ &= \varrho_l^*\big(\iota_l^*\big(\mathbf{Cl}_i\big(\gamma^n\big)\big)\big)\\ &= \varrho_l^*\big(\mathbf{Cl}_i\big(\gamma^n\big(\mathbb{F}^{n+l}\big)\big)\big)\\ &= \mathbf{Cl}_i(\xi). \qquad \square\end{aligned}$$

Now we use the characterisation of the classes $\mathbf{Cl}_i(\xi)$ given in Proposition 12 to generalise them for numerable vector bundles over any base space.

Recall that an open covering $\{U_i\}_{i\in\Lambda}$ of a space B is called *numerable* provided there exists a partition of unity $\{\phi_i\}_{i\in\Lambda}$ such that the support of ϕ_i is contained in U_i for each $i \in \Lambda$. An $\mathbb{F}$-vector bundle ξ over a space B is numerable provided that there is a numerable cover $\{U_i\}_{i\in\Lambda}$ of B such that $\xi|_{U_i}$ is trivial for each $i \in \Lambda$. By [12, Theorems 12.2 and 12.4] every numerable $\mathbb{F}$-vector bundle ξ of rank n has a classifying map $f_\xi : B \to G_n$, i.e. $\xi \cong f_\xi^*(\gamma^n)$ which is unique up to homotopy.

Theorem 13. *Let ξ be a numerable $\mathbb{F}$-vector bundle of rank n over a base space B. Let $\psi_\xi : B \to G_n$ be the classifying map of ξ. We define $\mathbf{Cl}_i(\xi) = \psi_\xi^*(\mathbf{Cl}_i(\gamma^n))$. Then $\mathbf{Cl}_i(\xi)$ satisfies Axioms* A1, A2, A3$'$ *and* A4$'$.

Proof. We just need to check Axioms A1, A2 and A3$'$, since by Proposition 12 the verification of Axiom 4$'$ is the same as in Theorem 11.

Axiom A1. By definition $\mathbf{Cl}_i(\xi) \in H^{bi}(B; K_b)$.

If $i = 0$ by Theorem 11 Axiom A1 $\mathbf{Cl}_0(\gamma^n(\mathbb{F}^{n+l})) = 1$ for all l, then by (9) $\mathbf{Cl}_0(\gamma^n) = 1$ and therefore $\mathbf{Cl}_0(\xi) = 1 \in H^0(B; K_b)$.

Analogously, if $i > n$ by Theorem 11 Axiom A1 $\mathbf{Cl}_i(\gamma^n(\mathbb{F}^{n+l})) = 0$ for all l, then by (9) $\mathbf{Cl}_i(\gamma^n) = 0$ and therefore $\mathbf{Cl}_i(\xi) = 0 \in H^{bi}(B; K_b)$.

Axiom A2. Let $f : B' \to B$ a continuous map. Let ψ_ξ and $\psi_{f^*\xi}$ be the classifying maps of ξ and $f^*\xi$ respectively. Since $f^*\xi = \psi_{f^*\xi}^*(\gamma^n) = (\psi_\xi \circ f)^*(\gamma^n)$, $\psi_{f^*\xi}$ and $\psi_\xi \circ f$ are homotopic [12, Theorem 12.4]. Hence

$$\begin{aligned}\psi_{f^*\xi}^*\big(\mathbf{Cl}_i\big(\gamma^n\big)\big) &= (\psi_\xi \circ f)^*\big(\mathbf{Cl}_i\big(\gamma^n\big)\big)\\ \mathbf{Cl}_i(f^*\xi) &= f^*\big(\psi_\xi^*\big(\mathbf{Cl}_i\big(\gamma^n\big)\big)\big)\\ &= f^*\big(\mathbf{Cl}_i(\xi)\big).\end{aligned}$$

Axiom A3$'$. Let ε_B^k be the trivial bundle of rank k over B. Let $\psi_\xi : B \to G_n$ and $\psi_{\xi\oplus\varepsilon_B^k} : B \to G_{n+k}$ be the classifying maps of ξ and $\xi \oplus \varepsilon_B^k$ respectively. Let $\ell_n : G_n \to G_{n+k}$ be the classifying map of the bundle $\gamma^n \oplus \varepsilon_{G_n}^k$. We have that

$$\begin{aligned}\psi_\xi^*\big(\gamma^n \oplus \varepsilon_{G_n}^k\big) &= \psi_\xi^*\big(\gamma^n\big) \oplus \psi_\xi^*\big(\varepsilon_{G_n}^k\big)\\ &= \xi \oplus \varepsilon_B^k.\end{aligned}$$

Thus $\xi \oplus \varepsilon_B^k = \psi_{\xi\oplus\varepsilon_B^k}^*(\gamma^{n+l}) = (\ell_n \circ \psi_\xi)^*(\gamma^{n+l})$ and therefore $\psi_{\xi\oplus\varepsilon_B^k}$ and $\ell_n \circ \psi_\xi$ are homotopic [12, Theorem 12.4]. Hence

$$\mathbf{Cl}_i\big(\xi \oplus \varepsilon_B^k\big) = \psi_\xi^*\big(\mathbf{Cl}_i\big(\gamma^n \oplus \varepsilon_{G_n}^k\big)\big). \tag{11}$$

We have the inclusions $\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l})}^k \subset \gamma^n(\mathbb{F}^{n+l+1}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l+1})}^k \subset \cdots$ and therefore $\gamma^n \oplus \varepsilon_{G_n}^k = \bigcup_k(\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l})}^k)$. By diagram (8) we have that $\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l})}^k = \iota^*(\gamma^n(\mathbb{F}^{n+l+1}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l+1})}^k)$ and since $\iota_l^*(\gamma^n) = \gamma^n(\mathbb{F}^{n+l})$ we also have that $\iota_l^*(\gamma^n \oplus \varepsilon^k_{G_n}) = \gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon^k_{G_n(\mathbb{F}^{n+l})}$. By Proposition 12 $\mathbf{Cl}_i(\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon^k_{G_n(\mathbb{F}^{n+l})}) = \iota_l^*(\ell_n^*(\mathbf{Cl}_i(\gamma^{n+k}))) = \iota_l^*(\mathbf{Cl}_i(\gamma^n \oplus \varepsilon^k_{G_n}))$ then

$$\mathbf{Cl}_i\big(\gamma^n \oplus \varepsilon^k_{G_n}\big) = \lambda^{-1}\big(\big(\mathbf{Cl}_i\big(\gamma^n\big(\mathbb{F}^n\big) \oplus \varepsilon^k_{G_n(\mathbb{F}^n)}\big), \ldots, \mathbf{Cl}_i\big(\gamma^n\big(\mathbb{F}^{n+l}\big) \oplus \varepsilon^k_{G_n(\mathbb{F}^{n+l})}\big), \ldots\big)\big).$$

But by Theorem 11 Axiom A3′ $\mathbf{Cl}_i(\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon^k_{G_n(\mathbb{F}^{n+l})}) = \mathbf{Cl}_i(\gamma^n(\mathbb{F}^{n+l}))$ for all l, hence by (9) $\mathbf{Cl}_i(\gamma^n \oplus \varepsilon^k_{G_n}) = \mathbf{Cl}_i(\gamma^n)$. Therefore by (11) $\mathbf{Cl}_i(\xi \oplus \varepsilon^k_B) = \mathbf{Cl}_i(\xi)$. □

8. Uniqueness

In this section we prove the uniqueness of the classes $\mathbf{Cl}_i(\xi)$.

Lemma 14. *Let $p : \gamma^k \to G_k$ be the universal bundle over G_k. Let p_0 be the restriction of p to the subspace $E(\gamma^k)_0$ of non-zero vectors of the total space $E(\gamma^k)$. Then $p_0^*(\gamma^k) = \eta^{k-1} \oplus \varepsilon^1$ and $\mathbf{Cl}_k(p_0^*(\gamma^k)) = 0$.*

Proof. We have that $p_0^*(\gamma^k) = \{(\ell, v, w) \mid \ell \in G_k,\ v, w \in \ell \text{ and } v \neq 0\}$. The map $s : E_0 \to p_0^*(\gamma^k)$ given by $s(\ell, v) = (\ell, v, v)$ is a non-vanishing global section. This section defines a trivial line subbundle $\varepsilon^1 \subset p_0^*(\gamma^k)$. Since $\mathbb{F}^\infty$ has an euclidean metric, there is a canonical Riemannian metric on γ^k, and we consider the pull-back of this metric on $p_0^*(\gamma^k)$. We define η^{k-1} to be its orthogonal complement. Therefore $p_0^*(\gamma^k) \cong \eta^{k-1} \oplus \varepsilon^1$ and by Theorem 13 we have that $\mathbf{Cl}_k(p_0^*(\gamma^k)) = 0$. □

Theorem 15. *Let ξ be a numerable $\mathbb{F}$-vector bundle of rank n. Then $\mathbf{Cl}_k(\xi) = \mathbf{cl}_k(\xi)$ for every k.*

Proof. Let $\psi_\xi : B \to G_n$ be the classifying map of ξ, that is $\xi = \psi_\xi^*(\gamma^n)$. Let $k \leqslant n$ and let $\ell_k : G_k \to G_n$ be the classifying map of the bundle $\gamma^k \oplus \varepsilon^{n-k}$. Axioms A2 and A3 imply Axiom A3′ [5, Proposition 3, p. 39] so for $i = 0, \ldots, n$ we have that

$$\begin{aligned} \ell_k^*\big(\mathbf{cl}_i\big(\gamma^n\big)\big) &= \mathbf{cl}_i\big(\ell_k^*\big(\gamma^n\big)\big) \\ &= \mathbf{cl}_i\big(\gamma^k \oplus \varepsilon^{n-k}\big) \\ &= \mathbf{cl}_i\big(\gamma^k\big). \end{aligned} \tag{12}$$

On the other hand, again by Axioms A2 and A3′ we have for $i = 0, \ldots, n$ that

$$\begin{aligned} \ell_k^*\big(\mathbf{Cl}_i\big(\gamma^n\big)\big) &= \mathbf{Cl}_i\big(\ell_k^*\big(\gamma^n\big)\big) \\ &= \mathbf{Cl}_i\big(\gamma^k \oplus \varepsilon^{n-k}\big) \\ &= \mathbf{Cl}_i\big(\gamma^k\big). \end{aligned}$$

By Lemma 14 $p_0^*(\mathbf{Cl}_k(\gamma^k)) = 0$ and by the Gysin exact sequence [5, Theorem 12.2] for the bundle $p_0^*(\gamma^k)$

$$H^0(G_k; K_b) \xrightarrow{\bigcup \mathbf{cl}_k(\gamma^k)} H^{bk}(G_k; K_b) \xrightarrow{p_0^*} H^{bk}(E(\gamma^k)_0; K_b) \to$$

there exists $\alpha_k \in H^0(G_k; K_b)$ such that $\mathbf{Cl}_k(\gamma^k) = \alpha_k \cup \mathbf{cl}_k(\gamma^k)$. Since G_n is path-connected for all n, ℓ_k induces an isomorphism in 0-dimensional cohomology. Let ρ_k be the unique element in $H^0(G_n; K_b)$ such that $\ell_k^*(\rho_k) = \alpha_k$. Hence

$$\begin{aligned} \ell_k^*\big(\mathbf{Cl}_k\big(\gamma^n\big)\big) &= \mathbf{Cl}_k\big(\gamma^k\big) \\ &= \alpha_k \cup \mathbf{cl}_k\big(\gamma^k\big) \\ &= \alpha_k \cup \ell_k^*\big(\mathbf{cl}_k\big(\gamma^n\big)\big) \\ &= \ell_k^*\big(\rho_k \cup \mathbf{cl}_k\big(\gamma^n\big)\big). \end{aligned}$$

The cohomology ring $H^*(G_n; K_b)$ is the polynomial ring $K_b[\mathbf{cl}_1(\gamma^n), \dots, \mathbf{cl}_n(\gamma^n)]$ on the Stiefel–Whitney classes of γ^n for $\mathbb{F} = \mathbb{R}$ [12, Theorem 20.5.2] or on the Chern classes of γ^n for $\mathbb{F} = \mathbb{C}$ [12, Theorem 20.3.2]. By (12) we have that the homomorphisms

$$H^{bi}(G_n; K_b) \xrightarrow{\ell_k^*} H^{bi}(G_k; K_b)$$

are isomorphisms for $i \leqslant k$. Thus

$$\mathbf{Cl}_k(\gamma^n) = \rho_k \cup \mathbf{cl}_k(\gamma^n).$$

Hence we have that

$$\begin{aligned}\mathbf{Cl}_k(\xi) &= \mathbf{Cl}_k(\psi_\xi^*(\gamma^n)) \\ &= \psi_\xi^*(\mathbf{Cl}_k(\gamma^n)) \\ &= \psi_\xi^*(\rho_k \cup \mathbf{cl}_k(\gamma^n)) \\ &= \psi_\xi^*(\rho_k) \cup \psi_\xi^*(\mathbf{cl}_k(\gamma^n)) \\ &= \beta_k \cup \mathbf{cl}_k(\psi_\xi^*(\gamma^n)) \\ &= \beta_k \cup \mathbf{cl}_k(\xi)\end{aligned}$$

with $\beta_k = \psi_\xi^*(\rho_k)$. The element $\beta_k \in H^0(B; K_b)$ is independent of the bundle ξ since for any path-connected space B and any map $f : B \to G_n$ the induced homomorphism $f^* : H^0(G_n; K_b) \to H^0(B; K_b)$ is the same isomorphism.

Let ζ^k be the canonical k-bundle over $\mathbb{F}\mathrm{P}^k$. By Axiom A4′ we have that $\mathbf{Cl}_k(\zeta^k) = (-1)^k g_k \in H^{bk}(\mathbb{F}\mathrm{P}^k; K_b)$.

Consider the class $\mathbf{cl}_1(\gamma_k^1) \in H^b(\mathbb{F}\mathrm{P}^k; K_b)$. Then by [5, p. 170] we have that

$$g_k = (-1)^k \mathbf{cl}_1(\gamma_k^1)^k. \tag{13}$$

By Lemma 1, ζ^k is the Whitney sum of k copies of the canonical line bundle γ_k^1 over $\mathbb{F}\mathrm{P}^k$. Using Axiom A3 and (13) we have that

$$\begin{aligned}\mathbf{cl}_k(\zeta^k) &= \mathbf{cl}_k(\gamma_k^1 \oplus \cdots \oplus \gamma_k^1) \\ &= \mathbf{cl}_1(\gamma_k^1)^k \\ &= (-1)^k g_k.\end{aligned}$$

Since $\mathbf{Cl}_k(\zeta^k) = \beta_k \cup \mathbf{cl}_k(\zeta^k)$, this implies that $\beta_k = 1$. Therefore $\mathbf{Cl}_k(\xi) = \mathbf{cl}_k(\xi)$ for every bundle ξ and any k. □

Corollary 16 *(Uniqueness). The classes* $\mathbf{Cl}_i(\xi)$ *for a smooth* $\mathbb{F}$*-vector bundle are well defined and hence they are well defined for any numerable* $\mathbb{F}$*-vector bundle.*

Proof. If we take in Theorem 11 a different generic vector bundle morphism $g : \varepsilon^{n-i+1} \to \xi$, the classes $\mathbf{Cl}_i^g(\xi) = \phi!([\tilde{Z}(g)])$ defined with it will also satisfy Theorem 15, so they coincide with the characteristic classes $\mathbf{cl}_i(\xi)$ which are unique [5, Theorem 73]. □

Corollary 17. *The classes* $\mathbf{Cl}_i(\xi)$ *are well defined for any* $\mathbb{F}$*-vector bundle* ξ *over a paracompact base space.*

Proof. Since any open cover of a paracompact space B has a partition of unity subordinated to a locally finite refinement, then any $\mathbb{F}$-vector bundle over B is numerable. □

Remark 18. One could also define the classes $\mathbf{Cl}_i(\xi)$ using the theory of manifolds with singularities of Borel and Haefliger. By results of [24] the closure of the singular set $\bar{Z}_1(\tau)$ given in (2) has a fundamental class in homology with coefficients in K_b with closed support: for the real case see [24, Theorem 3.7] and for the complex case [24, Theorem 3.2 and Proposition 2.7] (see also [25]). If $h : \varepsilon^{n-i+1} \to \xi$ is a generic bundle morphism by [24, Proposition 2.15] or [26, Lemme p. 8-02] $\bar{Z}_j(h)$ also has a fundamental class $[\bar{Z}_j(h)]$ in homology with coefficients in K_b with closed support. Then

$$\mathbf{Cl}_i(\xi) = \hat{J}\big(\big[\bar{Z}_1(h)\big]\big)$$

where $\hat{J} = D \circ J_*$ is the composition of the homomorphism induced in homology by the inclusion $J : \bar{Z}(h) \to M$ with the Poincaré duality isomorphism D.

Proposition 7 asserts that the map $\phi : \tilde{Z}(h) \to M$ is a desingularisation of the singular set $\bar{Z}_1(h)$ and by [24, Proposition 2.5] we have that the fundamental class of $\tilde{Z}(h)$ is mapped to the fundamental class of $\bar{Z}_j(h)$ by the homomorphism induced in homology by ϕ. Therefore both constructions coincide.

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